This is a Class 1 exam

Lebanon Evangelical School
for Boys and Girls
Loueizeh - Baabda

Stage Exam 05

07MAY2024

Do NOT write in this box. Your MARK will be placed here.

Subject :	Maths
Class :	11S
Teacher :	Taline Hawatian
Duration :	1 hour

Your name :		*Time taken to complete exam*
Your Class :		
Exam Room :		
How many sheets, including cover sheet :		

- Do not open this paper until the teacher in charge instructs you to do so.
- The most important thing to remember in taking an exam is NEVER PANIC.
- If you are caught cheating in this exam, you will be awarded zero for ALL exams that are taken on this day.

I.- (15 points)

Denote by (C_m) the set of points $M(x;y)$ such that: $x^2 + y^2 - 2mx - 2(m-1)y + 6m - 9 = 0$.

1) Show that, for all m, (C_m) is a circle whose center I_m and radius R are to be determined.
2) Deduce that I_m varies on a fixed straight line to be determined.
3) Show that all the circles pass through two fixed points to be determined.
4) Consider the circle (C_1) of equation $x^2 + y^2 - 2x - 3 = 0$.
   a) Determine the center I_1 and radius R of (C_1).
   b) Write the equations of the tangents to (C_1) through the point A(-2, 3).
5) Solve, graphically, the following system of inequalities:

$$\begin{cases} x^2 + y^2 - 2x - 3 \leq 0 \\ 2x - 3y + 3 \geq 0 \end{cases}$$

II.- (7 points)

1) Without using a calculator, calculate $\left[\frac{\sqrt{2}}{2}(1+i)\right]^2$ and $\left[\frac{\sqrt{2}}{2}(1-i)\right]^2$. Deduce the square roots of i and $-i$.

   Solve the equation $z^4 + 1 = 0$.
2) Solve in C the equation $\bar{z} + 2iz = \frac{1-i}{2+i}$.

III.- (8 points)

Given the complex number $z = x + iy$, where x and y are real numbers, and the complex number Z' defined by $Z' = \frac{iz+1}{z-1}$. M$(x;y)$ is a point of the complex plane of affix z.

1) Write the algebraic form of Z'.
2) Calculate, in terms of x and y, Re(Z') and Im(Z').
3) Find the set of points M for which:
   a) Z' is pure imaginary.
   b) Z' is real.

SE05/20232024/Maths/11S

I.- 1) $x^2+y^2-2mx-2(m-1)y+6m-9=0$

$x^2-2mx+m^2+y^2-2(m-1)y+(m-1)^2=m^2+(m-1)^2-6m+9$

$(x-m)^2+[y-(m-1)]^2=2m^2-8m+10$ 1

$2m^2-8m+10=0 \quad \Delta'=16-20=-4<0$ 1

$\Delta'<0$ and $2>0 \Rightarrow 2m^2-8m+10>0$ for all m in $\mathbb{R}$.

$\Rightarrow (C_m)$ is a circle of center $I_m(m,m-1)$ and radius $R=\sqrt{2m^2-8m+10}$. 1

2) $I_m(m,m-1) \quad x=m \quad y=m-1 \quad y=x-1$ 1

I_m varies on the straight line of equation $(d): y=x-1$.

3) $x^2+y^2-2mx-2my+2y+6m-9=0$

$x^2+y^2+2y-9+(-2x-2y+6)m=0$

$\begin{cases} x^2+y^2+2y-9=0 \\ (-2x-2y+6=0) \ (\div -2) \end{cases}$ 1 $\quad x+y-3=0 \quad y=3-x$

$x^2+(3-x)^2+2(3-x)-9=0$ 1

$x^2+9-6x+x^2+6-2x-9=0 \quad (2x^2-8x+6=0)(\div 2)$

$x^2-4x+3=0 \quad x=1 \quad y=3-1=2$

$x=3 \quad y=0$ 1

The fixed points are $(1;2)$ and $(3;0)$.

4) $(C_1): x^2+y^2-2x-3=0$ 1

a) $I_m(m,m-1) \quad I_1(1;0) \quad R=\sqrt{2-8+10}=\sqrt{4}=2$

(C_1) is the circle of center $I(1;0)$ and radius $R=2$.

b) $A(-2;3)$ Let T be the tangent drawn through A to (C).

$(T): y-3=a(x+2) \quad y-3=ax+2a \quad (T): ax-y+2a+3=0$ 0.5

$d[I/(T)]=R \quad \dfrac{|m-0+2m+3|}{\sqrt{m^2+1}}=2$ 0.5

$|3m+3|=2\sqrt{m^2+1}$ square both sides

$(3m+3)^2 = 4(m^2+1)$ 1

$9m^2+18m+9-4m^2-4=0 \qquad 5m^2+18m+5=0$

$\Delta' = 81-25 = 56 \qquad m_1, m_2 = \frac{-9 \pm 2\sqrt{14}}{5}$ 0.5

(T): $y-3 = \left(\frac{-9 \pm 2\sqrt{14}}{5}\right)(x+2)$ 0.5

5) $\begin{cases} x^2+y^2-2x-3 \leq 0 \quad \ldots (1) \\ 2x-3y+3 \geqslant 0 \quad \ldots (2) \end{cases}$

(1): $x^2+y^2-2x-3=0 \quad (x-1)^2+y^2=4 \quad I(1;0) \quad R=2$

(2): $2x-3y+3=0$

x	0	-3/2	1
y	1	0	5/3

$O(0;0) \quad 2x-3y+3 = 2(0)-3(0)+3 = 3 > 0$ 0.5

$O(0;0)$ is in the solution region. 0.5

The solution region is the double-shaded region (S) including the straight line and the periphery of the disc.

II.- 1) $\left[\frac{\sqrt{2}}{2}(1+i)\right]^2 = \frac{2}{4}(1+i)^2 = \frac{1}{2}(1+2i+i^2) = \frac{1}{2}(1+2i-1) = i$ 1

$\left[\frac{\sqrt{2}}{2}(1-i)\right]^2 = \frac{2}{4}(1-i)^2 = \frac{1}{2}(1-2i+i^2) = -i$ 1

$i = \left[\frac{\sqrt{2}}{2}(1+i)\right]^2 \qquad \sqrt{i} = \pm\frac{\sqrt{2}}{2}(1+i)$

$-i = \left[\frac{\sqrt{2}}{2}(1+i)\right]^2 \qquad \sqrt{-i} = \pm\frac{\sqrt{2}}{2}(1-i)$ 0.5

$z^4+1=0 \qquad z^4 = -1 = i^2$

$z^2 = i$ and $z^2 = -i$

$z = \pm\sqrt{i} \qquad z = \pm\sqrt{-i}$ 1

$z_1 = \frac{\sqrt{2}}{2}(1+i) \quad z_2 = -\frac{\sqrt{2}}{2}(1+i) \quad z_3 = \frac{\sqrt{2}}{2}(1-i) \quad z_4 = -\frac{\sqrt{2}}{2}(1-i)$

2) $\bar{z} + 2iz = \frac{1-i}{2+i}$ let $z = x+iy$

$$x - iy + 2i(x+iy) = \frac{1-i}{2+i} \cdot \frac{2-i}{2-i}$$

$$x - iy + 2xi + 2yi^2 = \frac{2 - i - 2i + i^2}{4 - i^2}$$ 1

$$(x-2y) + (2x-y)i = \frac{1-3i}{5}$$

$$(x-2y) + (2x-y)i = \frac{1}{5} - \frac{3}{5}i$$ 1

$$\begin{cases} x - 2y = \frac{1}{5} \\ (2x - y = -\frac{3}{5})(-2) \end{cases} \quad \begin{cases} x - 2y = \frac{1}{5} \\ -4x + 2y = \frac{6}{5} \end{cases}$$ 1

$$-3x = \frac{7}{5} \quad x = \frac{-7}{15} \quad y = \frac{-1}{3}$$

$$z = \frac{-7}{15} - \frac{1}{3}i$$ 0.5

III.- 1) $z' = \frac{iz+1}{z-1}$ $\qquad z = x + iy$

$$= \frac{i(x+iy)+1}{x+iy-1}$$

$$= \frac{ix + i^2y + 1}{(x-1)+iy} \cdot \frac{(x-1)-iy}{(x-1)-iy}$$ 0.5

$$= \frac{(1-y)+ix}{(x-1)+iy} \cdot \frac{(x-1)+iy}{(x-1)+iy}$$

$$= \frac{(1-y)(x-1) + iy(1-y) + ix(x-1) + i^2xy}{(x-1)^2 + i^2y^2}$$ 1

$$= \frac{x - 1 - xy + y - iy + iy^2 + ix^2 - ix + xy}{(x-1)^2 + y^2}$$

$$= \frac{x+y-1}{(x-1)^2+y^2} + \frac{(x^2 - x + y^2 - y)}{(x-1)^2+y^2}$$ 1.5

2) $Re(z') = \frac{x+y-1}{(x-1)^2+y^2}$ 0.5 $\qquad Im(z') = \frac{x^2-x+y^2-y}{(x-1)^2+y^2}$ 0.5

3) a) Z' is pure imaginary.

$Re(Z') = 0$ and $Im(Z') \neq 0$

$\frac{x+y-1}{(x-1)^2+y^2} = 0$ $\quad x+y-1=0$ $\quad (x-1)^2+y^2 \neq 0$

$y = -x+1$ $\quad (1;0)$

$\frac{x^2-x+y^2-y}{(x-1)^2+y^2} \neq 0$ $\quad x^2-x+y^2-y \neq 0$

$x^2-x+(-x+1)^2+x-1=0$

$x^2-x+x^2-2x+1+x-1=0$

$2x^2-2x=0$

$2x(x-1)=0$ $\quad x=0$ $\quad y=1$

$x=1$ $\quad y=0$. 1.5

The set of points M is the straight line (d) of equation $x+y-1=0$ excluding the points $(0;1)$ and $(1;0)$.

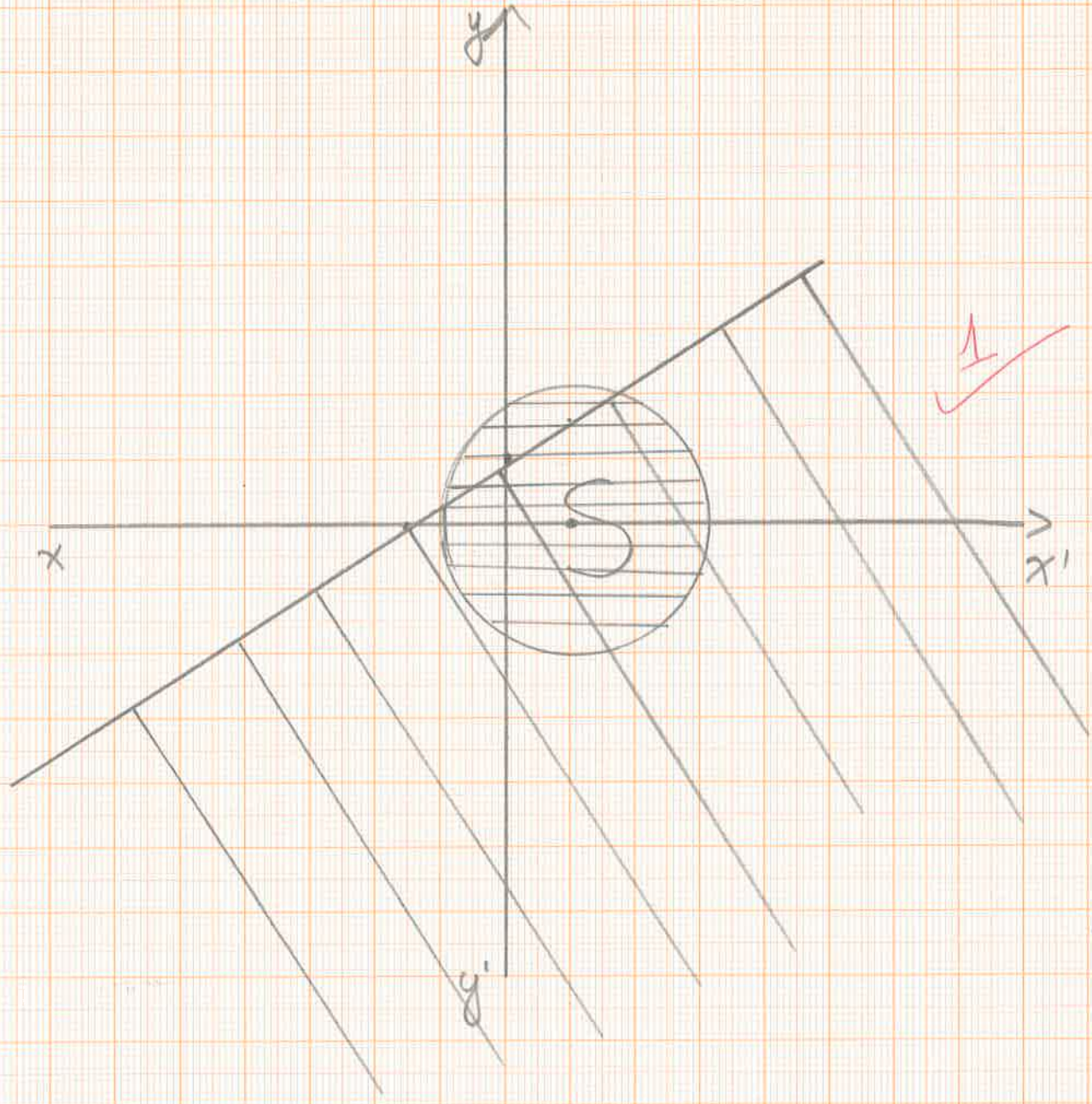
b) $Im(Z') = 0$

$x^2-x+y^2-y=0$ and $(x-1)^2+y^2 \neq 0$

$x^2-x+\frac{1}{4}+y^2-y+\frac{1}{4} = \frac{1}{4}+\frac{1}{4}$

$(x-\frac{1}{2})^2+(y-\frac{1}{2})^2 = \frac{1}{2}$

The set of points M is the circle of center $I(\frac{1}{2};\frac{1}{2})$ and radius $R=\frac{\sqrt{2}}{2}$ excluding the point $(1;0)$. 2



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